Models of Set Theory II - Winter 2013

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Problem 13 (4 Points). Suppose that M is a ground model. Let $(P, \leq_P, 1_P)$ and $(Q, \leq_Q, 1_Q)$ denote partial orders. Suppose that $\pi\colon P \to Q$ is a complete embedding in M (see Problems 22 and 23 from Models of Set Theory I). We define a map $\pi^*\colon V^P \to V^Q$ by recursion on $\tau \in V^P$:

$$\pi^*(\tau) = \{(\pi^*(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}.$$

Suppose that H is M-generic for Q. Then its preimage $G := \pi^{-1}[H]$ is M-generic for P (by Problem 23 from Models of Set Theory). Show that $\tau^G = \pi^*(\tau)^H$ for all $\tau \in V^P$.

Problem 14 (8 Points). Suppose that M is a ground model. Let $(P, \leq_P, 1_P)$ and $(Q, \leq_Q, 1_Q)$ denote partial orders in M. A *projection* from Q to P is a map $\pi\colon Q \to P$ such that

(1) $\pi(p) \leq_P \pi(q)$ if $p \leq_Q q$,
(2) $\pi(1_Q) = 1_P$, and
(3) for all $q \in Q$ and all $p \leq \pi(q)$, there is some $\bar{q} \leq q$ such that $\pi(\bar{q}) \leq p$.

Suppose that $\pi\colon P \to Q$ is a projection from Q to P in M. Prove the following statements.

(a) If H is an M-generic filter for Q, then the pointwise image of H under π generates an M-generic filter on P.
(b) Suppose that G is M-generic for P and let $Q/G = \{q \in Q \mid \pi(q) \in G\}$ with the partial order inherited from Q. If H is $M[G]$-generic for Q/G, then H is M-generic for Q.

Problem 15 (10 Points). Suppose that M is a ground model. Let $(P, \leq_P, 1_P)$ and $(Q, \leq_Q, 1_Q)$ denote partial orders in M. Work in M.

(a) Let $\check{Q}$ denote the canonical P-name for Q. Show that there is a dense embedding $\pi\colon P \times Q \to P * \check{Q}$.
(b) Show that $P \times Q$ is c.c.c if and only if P is c.c.c. and $1_P \Vdash_P \check{Q}$ is c.c.c.
(c) A *Suslin tree* is a (downwards closed) subtree $(T, \subseteq)$ of $({}^{<\omega_1}2, \subseteq)$ of height ω_1 such that
    (i) $(T, \subseteq)$ has no uncountable branches and
    (ii) $(T, \leq_T)$ has no uncountable antichains,

    where $p \leq_T q :\Longleftrightarrow p \supseteq q$ for $p, q \in T$. Show that if there is a Suslin tree, then there is a Suslin tree T such that for every node $s \in T$, there are uncountably many nodes $t \in T$ with $s \subseteq t$.

(d) Suppose that T is a Suslin tree. Show that the product of $(T, \leq_T)$ with $(T, \leq_T)$ is not c.c.c.

(e) Prove from MA_{ω_1} that there are no Suslin trees.

Please hand in your solutions on Monday, November 18 before the lecture.